

G22.2390-001 Logic in Computer Science
Fall 2009
Lecture 2

Review

Last week

- Propositional Logic: Syntax
- Well-Formed Formulas (*wffs*)
- Induction and Recursion

Outline

- Recognizing Well-Formed Formulas
- Propositional Logic: Semantics
- Truth Tables
- Satisfiability and Tautologies
- Propositional Connectives and Boolean Functions
- Compactness
- Computability and Decidability

Sources:

Enderton, Sections: 1.2, 1.3, 1.5, 1.7.

N. J. Cutland. *Computability*.

W. Hodges. *A Shorter Model Theory*. Cambridge Press, 1997.

Propositional Logic: Well-Formed Formulas

Recall our inductive definition of the set W of well-formed formulas in propositional logic. Given the alphabet $\{(,), \neg, \wedge, \vee, \rightarrow, \leftrightarrow, A_1, A_2, \dots\}$,

- $U =$
- $B =$
- $F =$

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- U = the set of all expressions over the alphabet.
- B = the set of expressions consisting of a single propositional symbol.
- F = the set of formula-building operations:
 - $\mathcal{E}_{\neg}(\alpha) = (\neg\alpha)$
 - $\mathcal{E}_{\wedge}(\alpha, \beta) = (\alpha \wedge \beta)$
 - $\mathcal{E}_{\vee}(\alpha, \beta) = (\alpha \vee \beta)$
 - $\mathcal{E}_{\rightarrow}(\alpha, \beta) = (\alpha \rightarrow \beta)$
 - $\mathcal{E}_{\leftrightarrow}(\alpha, \beta) = (\alpha \leftrightarrow \beta)$

An Algorithm for Recognizing WFFs

Lemma

Let α be a *wff*. Then exactly one of the following is true.

- α is a propositional symbol.
- $\alpha = (\neg\beta)$ where β is a *wff*.
- $\alpha = (\beta \odot \gamma)$ where \odot is one of $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$, β is the first parentheses-balanced initial segment of the result of dropping the first $($ from α , and β and γ are *wffs*.

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How would you prove this?

Induction, of course!

An Algorithm for Recognizing WFFs

Input: expression α Output: *true* or *false* (indicating whether α is a *wff*).

0. Begin with an initial construction tree T containing a single node labeled with α .
1. If all leaves of T are labeled with propositional symbols, return *true*.
2. Select a leaf labeled with an expression α_1 which is not a propositional symbol.
3. If α_1 does not begin with (return *false*.
4. If $\alpha_1 = (\neg\beta)$, then add a child to the leaf labeled by α_1 , label it with β , and goto 1.
5. Scan α_1 until first reaching β , where β is a nonempty expression having the same number of left and right parentheses. If there is no such β , return *false*.
6. If $\alpha_1 = (\beta \odot \gamma)$ where \odot is one of $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then add two children to the leaf labeled by α_1 , label them with β and γ , and goto 1.
7. Return *false*.

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Soundness

If the algorithm returns *true* when given input α , then α is a *wff*.

The proof is by induction on the tree T generated by the algorithm from the leaves up to the root.

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Completeness

If α is a *wff*, then the algorithm will return *true*.

Proof using the induction principle for the set of *wffs*.

Notational Conventions

- Larger variety of propositional symbols: A, B, C, D, p, q, r , etc.
- Outermost parentheses can be omitted: $A \wedge B$ instead of $(A \wedge B)$.
- Negation symbol binds stronger than binary connectives and its scope is as small as possible: $\neg A \wedge B$ means $((\neg A) \wedge B)$.
- $\{\wedge, \vee\}$ bind stronger than $\{\rightarrow, \leftrightarrow\}$: $A \wedge B \rightarrow \neg C \vee D$ is $((A \wedge B) \rightarrow ((\neg C) \vee D))$
- When one symbol is used repeatedly, grouping is to the right: $A \wedge B \wedge C$ is $(A \wedge (B \wedge C))$

Note that conventions are only unambiguous for *wffs*, not for arbitrary expressions.

Propositional Logic: Semantics

Intuitively, given a *wff* α and a value (either **T** or **F**) for each propositional symbol in α , we should be able to determine the value of α .

How do we make this precise?

Let v be a function from B to $\{\mathbf{F}, \mathbf{T}\}$. We call this function a *truth assignment*.

Now, we define \bar{v} , a function from W to $\{\mathbf{F}, \mathbf{T}\}$ as follows (we compute with **F** and **T** as if they were **0** and **1** respectively).

- For each propositional symbol A_i , $\bar{v}(A_i) = v(A_i)$.
- $\bar{v}(\mathcal{E}_{\neg}(\alpha)) = \mathbf{T} - \bar{v}(\alpha)$
- $\bar{v}(\mathcal{E}_{\wedge}(\alpha, \beta)) = \min(\bar{v}(\alpha), \bar{v}(\beta))$
- $\bar{v}(\mathcal{E}_{\vee}(\alpha, \beta)) = \max(\bar{v}(\alpha), \bar{v}(\beta))$
- $\bar{v}(\mathcal{E}_{\rightarrow}(\alpha, \beta)) = \max(\mathbf{T} - \bar{v}(\alpha), \bar{v}(\beta))$
- $\bar{v}(\mathcal{E}_{\leftrightarrow}(\alpha, \beta)) = \mathbf{T} - |\bar{v}(\alpha) - \bar{v}(\beta)|$

The recursion theorem and the unique readability theorem guarantee that \bar{v} is well-defined.

Truth Tables

There are other ways to present the semantics which are less formal but perhaps more intuitive.

α	$\neg\alpha$
T	
F	

α	β	$\alpha \wedge \beta$
T	T	
T	F	
F	T	
F	F	

α	β	$\alpha \vee \beta$
T	T	
T	F	
F	T	
F	F	

α	β	$\alpha \rightarrow \beta$
T	T	
T	F	
F	T	
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α	β	$\alpha \leftrightarrow \beta$
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T	T	T
T	F	F
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Complex truth tables

Truth tables can also be used to calculate all possible values of \bar{v} for a given *wff*: We associate a column with each propositional symbol and a column with each propositional connective. There is a row for each possible truth assignment to the propositional connectives.

A_1	A_2	A_3	$(A_1 \vee (A_2 \wedge \neg A_3))$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
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- If $\alpha \models \beta$ (shorthand for $\{\alpha\} \models \beta$) and $\beta \models \alpha$, then α and β are *tautologically equivalent*.
- $\Sigma \models \alpha$ if and only if $\bigwedge(\Sigma) \rightarrow \alpha$ is valid.

Examples

- $(A \vee B) \wedge (\neg A \vee \neg B)$

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Now suppose you had an algorithm *CHECKVALID* which returns *true* when α is valid and *false* otherwise. How would you verify the claims given this algorithm?

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- $\{A, \neg A\} \models (A \wedge \neg A)$ $(A \wedge (\neg A) \wedge \neg(A \wedge \neg A))$
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Now suppose you had an algorithm *CHECKVALID* which returns *true* when α is valid and *false* otherwise. How would you verify the claims given this algorithm?

Satisfiability and validity are dual notions: α is unsatisfiable if and only if $\neg\alpha$ is valid.

Determining Satisfiability using Truth Tables

An Algorithm for Satisfiability

To check whether α is satisfiable, form the truth table for α . If there is a row in which **T** appears as the value for α , then α is satisfiable. Otherwise, α is unsatisfiable.

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An Algorithm for Tautological Implication

To check whether $\{\alpha_1, \dots, \alpha_k\} \models \beta$, check the satisfiability of $(\alpha_1 \wedge \dots \wedge \alpha_k) \wedge (\neg\beta)$. If it is unsatisfiable, then $\{\alpha_1, \dots, \alpha_k\} \models \beta$, otherwise $\{\alpha_1, \dots, \alpha_k\} \not\models \beta$.

Determining Satisfiability using Truth Tables

Example

$$A \wedge ((B \vee \neg A) \wedge (C \vee \neg B))$$

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A	B	C	$A \wedge ((B \vee \neg A) \wedge (C \vee \neg B))$

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Example

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A	B	C	$A \wedge ((B \vee \neg A) \wedge (C \vee \neg B))$
F	F	F	F

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F	F	F	F
F	F	T	F

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A	B	C	$A \wedge ((B \vee \neg A) \wedge (C \vee \neg B))$
F	F	F	F
F	F	T	F
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F	F	T	F
F	T	F	F
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F	F	T	F
F	T	F	F
F	T	T	F
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T	F	T	F

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F	F	F	F
F	F	T	F
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A	B	C	$A \wedge ((B \vee \neg A) \wedge (C \vee \neg B))$
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	F
T	F	F	F
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Can you think of a way to speed up these algorithms?

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In an upcoming lecture, we will discuss some of the applications and best-known techniques for the *SAT* algorithm.

Some tautologies

Associative and Commutative laws for $\wedge, \vee, \leftrightarrow$

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Distributive Laws

- $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)).$
- $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C)).$

Some tautologies

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Distributive Laws

- $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)).$
- $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C)).$

Negation

- $\neg\neg A \leftrightarrow A$
- $\neg(A \rightarrow B) \leftrightarrow (A \wedge \neg B)$
- $\neg(A \leftrightarrow B) \leftrightarrow ((A \wedge \neg B) \vee (\neg A \wedge B))$

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Associative and Commutative laws for $\wedge, \vee, \leftrightarrow$

Distributive Laws

- $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)).$
- $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C)).$

Negation

- $\neg\neg A \leftrightarrow A$
- $\neg(A \rightarrow B) \leftrightarrow (A \wedge \neg B)$
- $\neg(A \leftrightarrow B) \leftrightarrow ((A \wedge \neg B) \vee (\neg A \wedge B))$

De Morgan's Laws

- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

More Tautologies

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- $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$

Exportation

- $((A \wedge B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C))$

Propositional Connectives

We have five connectives: \neg , \wedge , \vee , \rightarrow , \leftrightarrow . Would we gain anything by having more? Would we lose anything by having fewer?

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Example: Ternary Majority Connective $\#$

$$\mathcal{E}_{\#}(\alpha, \beta, \gamma) = (\#\alpha\beta\gamma)$$

$\bar{v}(\#\alpha\beta\gamma) = \mathbf{T}$ iff the majority of $\bar{v}(\alpha)$, $\bar{v}(\beta)$, and $\bar{v}(\gamma)$ are \mathbf{T} .

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How do we show this formally?

Boolean Functions

For $k \geq 0$, a k -place Boolean function is a function from $\{\mathbf{F}, \mathbf{T}\}^k$ to $\{\mathbf{F}, \mathbf{T}\}$. A Boolean function then is anything which is a k -place Boolean function for some k .

Each wff α determines a corresponding Boolean function B_α . For example, if $\alpha = A_1 \wedge A_2$, then B_α is a 2-place Boolean function whose value is given by the following table.

X_1	X_2	$B_\alpha(X_1, X_2)$
T	T	T
T	F	F
F	T	F
F	F	F

Realizing Boolean Functions

In general, suppose that α is a *wff* whose propositional symbols are included in A_1, \dots, A_n . We define an n -place Boolean function B_α^n , the Boolean function *realized* by α as

$B_\alpha^n(X_1, \dots, X_n)$ = the truth value given to α when A_1, \dots, A_n are given the values X_1, \dots, X_n .

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Note that the function B_α^n is determined by *both* the formula α and the choice of n . In particular, α does not need to include all the symbols in A_1, \dots, A_n .

Examples

- $I_i^n = B_{A_i}^n$
- $N = B_{\neg A_1}^1$
- $K = B_{A_1 \wedge A_2}^2$
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From these functions, we can construct others by composition.

$$B_{\neg A_1 \vee \neg A_2}^2(X_1, X_2) = A(N(I_1^2(X_1, X_2)), N(I_2^2(X_1, X_2)))$$

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Claim: Every Boolean function can be obtained as a composition of I , N , K , A , C , and E .

We will explain why this is true shortly.

Formulas and the Boolean Functions they Realize

Theorem

Let α and β be *wffs* whose sentence symbols are among A_1, \dots, A_n .

- (a) $\alpha \models \beta$ iff $B_\alpha^n(\vec{X}) \leq B_\beta^n(\vec{X})$ for all $\vec{X} \in \{\mathbf{F}, \mathbf{T}\}^n$.
- (b) α is tautologically equivalent to β iff $B_\alpha^n = B_\beta^n$.
- (c) $\models \beta$ iff the range of $B_\beta^n = \{\mathbf{T}\}$.

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Proof

(a)

- $\alpha \models \beta$ iff every truth assignment satisfying α also satisfies β
- iff for every truth assignment v , $\bar{v}(\alpha) = \mathbf{T}$ implies $\bar{v}(\beta) = \mathbf{T}$
- iff for all n -tuples \vec{X} , $B_\alpha^n(\vec{X}) = \mathbf{T}$ implies $B_\beta^n(\vec{X}) = \mathbf{T}$
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(b) Follows from (a) and $X = Y$ iff $X \leq Y$ and $Y \leq X$.

(c) Follows from definition of tautology. □

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By shifting our focus from formulas to Boolean functions, tautologically equivalent *wffs* are identified.

Completeness of Propositional Connectives

Theorem (Post 1921)

Let G be an n -place Boolean function, $n \geq 1$. There exists a *wff* α such that $G = B_{\alpha}^n$, i.e., such that α realizes the function G .

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Proof

If the range of G is just $\{\mathbf{F}\}$, then let $\alpha = A_1 \wedge \neg A_1$. Clearly, $B_\alpha^n = G$.
Otherwise, $G = \mathbf{T}$ somewhere. Suppose there are k points where $G = \mathbf{T}$:

$$G(X_{11}, X_{12}, \dots, X_{1n}) = \mathbf{T}$$

$$G(X_{21}, X_{22}, \dots, X_{2n}) = \mathbf{T}$$

...

$$G(X_{k1}, X_{k2}, \dots, X_{kn}) = \mathbf{T}$$

$$\text{Let } \beta_{ij} = \begin{cases} A_j & \text{if } X_{ij} = \mathbf{T} \\ \neg A_j & \text{if } X_{ij} = \mathbf{F} \end{cases}$$
$$\gamma_i = \beta_{i1} \wedge \dots \wedge \beta_{in}$$
$$\alpha = \gamma_1 \vee \gamma_2 \vee \dots \vee \gamma_k$$

Then α realizes G

Completeness of Propositional Connectives

Proof, continued

We know that $B_{\alpha}^n(\vec{X}) = \bar{v}(\alpha)$ where $v(A_i) = X_i$.

Since $\alpha = \gamma_1 \vee \gamma_2 \vee \dots \vee \gamma_k$, it follows that $B_{\alpha}^n(\vec{X}) = \max(B_{\gamma_i}^n(\vec{X}))$.

But by construction, $B_{\gamma_i}^n(\vec{X}) = \mathbf{T}$ iff $\vec{X} = \langle X_{i1}, \dots, X_{in} \rangle$.

Thus $B_{\alpha}^n(\vec{X}) = \mathbf{T}$ iff \vec{X} is one of the points where G is \mathbf{T} .

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□

This shows that every Boolean function can be realized by a *wff*. In fact, every Boolean function can be realized by a *wff* which uses only the connectives $\{\neg, \wedge, \vee\}$. We say that this set of connectives is *complete*.

The realizing formula is not unique. The formula built is in so-called *disjunctive normal form* (DNF). A formula is in DNF if it is a disjunction of formulas, each of which is a conjunction of *literals*, where a literal is either a propositional symbol or its negation.

Thus, a corollary is that for every *wff*, there exists a tautologically equivalent *wff* in disjunctive normal form.

Completeness of Propositional Connectives

Example

Let G be a 3-place Boolean function defined as follows:

$$G(\mathbf{F}, \mathbf{F}, \mathbf{F}) = \mathbf{F}$$

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There are four points at which G is true, so a DNF formula which realizes G is

$$(\neg A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge A_2 \wedge \neg A_3) \vee (A_1 \wedge \neg A_2 \wedge \neg A_3) \vee (A_1 \wedge A_2 \wedge A_3).$$

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Note that another formula which realizes G is $A_1 \leftrightarrow A_2 \leftrightarrow A_3$. Thus, adding additional connectives to a complete set may allow a function to be realized more concisely.

Completeness of Propositional Connectives

Recall our definition of some basic Boolean functions:

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- $N = B_{\neg A_1}^1$
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Given that $\{\neg, \wedge, \vee\}$ is complete, it is not hard to see that any Boolean function can be constructed using only the Boolean functions I , N , K , and A .

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$$\alpha \vee \beta \leftrightarrow \neg(\neg\alpha \wedge \neg\beta)$$

$$\alpha \wedge \beta \leftrightarrow \neg(\neg\alpha \vee \neg\beta)$$

Using these identities, the completeness can be easily proved by induction.

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To prove that some set of connectives is incomplete, we find a property that is true of all *wffs* built using those connectives, but that is not true for some Boolean function.

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Thus, $\bar{v}(\alpha) = \mathbf{T}$ for all *wffs* α built from $\{\wedge, \rightarrow\}$. But $\bar{v}(\neg A_1) = \mathbf{F}$, so there is no such formula tautologically equivalent to $\neg A_1$. \square

Other Propositional Connectives

For each n , there are 2^{2^n} different n -place Boolean functions $B(X_1, \dots, X_n)$

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0-ary connectives

There are two 0-place Boolean functions: the constants **F** and **T**. We can construct corresponding 0-ary connectives \perp and \top with the meaning that $\bar{v}(\perp) = \mathbf{F}$ and $\bar{v}(\top) = \mathbf{T}$ regardless of the truth assignment v .

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There are four 1-place functions, but these include the two constant functions mentioned above and the identity function. Thus the only additional connective of interest is negation: \neg .

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Binary connectives

There are sixteen 2-place Boolean functions. They are cataloged in the following table. Note that the first six correspond to 0-ary and unary connectives.

Symbol	Equivalent	Description
	\perp	constant F
	\top	constant T
	A	projection of first argument
	B	projection of second argument
	$\neg A$	negation of first argument
	$\neg B$	negation of second argument
\wedge	$A \wedge B$	and
\vee	$A \vee B$	or
\rightarrow	$A \rightarrow B$	conditional
\leftrightarrow	$A \leftrightarrow B$	bi-conditional
\leftarrow	$B \rightarrow A$	reverse conditional
\oplus	$(A \wedge \neg B) \vee (\neg A \wedge B)$	exclusive or
\downarrow	$\neg(A \vee B)$	nor (or Nicod stroke)
\mid	$\neg(A \wedge B)$	nand (or Sheffer stroke)
$<$	$\neg A \wedge B$	less than
$>$	$A \wedge \neg B$	greater than

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A set Σ is *finitely satisfiable* iff every finite subset of Σ is satisfiable.

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Recall that a *wff* α is satisfiable if there exists a truth assignment v such that $\bar{v}(\alpha) = \mathbf{T}$.

A set Σ of *wffs* is satisfiable if there exists a truth assignment v such that $\bar{v}(\alpha) = \mathbf{T}$ for each $\alpha \in \Sigma$.

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To prove the other direction, assume that Σ is a set which is finitely satisfiable. We must show that Σ is satisfiable.

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Let Σ be finitely satisfiable. We extend Σ to form a *maximal* finitely satisfiable set Δ as follows.

Let $\alpha_1, \dots, \alpha_n, \dots$ be a fixed enumeration of all *wffs*.

Why is this possible?

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$$\begin{array}{l} \text{Then, let } \Delta_0 = \Sigma, \\ \Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if this is finitely satisfiable,} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise.} \end{cases} \end{array}$$

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It is not hard to show that each Δ_n is finitely satisfiable.

Let $\Delta = \bigcup_n \Delta_n$. It is then clear that

1. $\Sigma \subseteq \Delta$
2. $\alpha \in \Delta$ or $\neg\alpha \in \Delta$ for any *wff* α , and
3. Δ is finitely satisfiable.

Compactness

Now we show that Δ is satisfiable (and thus $\Sigma \subseteq \Delta$ is also satisfiable).

Define a truth assignment v as follows. For each propositional symbol A_i ,

$$v(A_i) = \mathbf{T} \text{ iff } A_i \in \Delta.$$

We claim that for any *wff* α , v satisfies α iff $\alpha \in \Delta$. The proof is by induction on well-formed formulas.

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Base Case

Follows directly from the definition of v .

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Induction Case

We will just consider one case. Suppose $\alpha = \beta \wedge \gamma$. Then

$$\bar{v}(\alpha) = \mathbf{T} \text{ iff both } \bar{v}(\beta) = \mathbf{T} \text{ and } \bar{v}(\gamma) = \mathbf{T} \text{ iff both } \beta \in \Delta \text{ and } \gamma \in \Delta.$$

Now, if both β and γ are in Δ , then since $\{\beta, \gamma, \neg\alpha\}$ is not satisfiable, we must have $\alpha \in \Delta$.

Similarly, if one of β or γ is not in Δ , then its negation must be in Δ , so $\alpha \notin \Delta$. \square

Compactness

Corollary

If $\Sigma \models \alpha$ then there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \alpha$.

Proof

Suppose that $\Sigma_0 \not\models \alpha$ for every finite $\Sigma_0 \subseteq \Sigma$.

Then, $\Sigma_0 \cup \{\neg\alpha\}$ is satisfiable for every finite $\Sigma_0 \subseteq \Sigma$.

So, by compactness, $\Sigma \cup \{\neg\alpha\}$ is satisfiable which contradicts the fact that $\Sigma \models \alpha$.

□