

G22.2390-001 Logic in Computer Science
Fall 2009
Lecture 7

Review

Last time

- Homomorphisms
- Completeness

Outline

- Compactness
- Enumerability Theorem
- Definability of Models
- Finite Models
- Size of Models

Sources:

Sections 2.5 through 2.7 of Enderton.

W. Hodges. [*A Shorter Model Theory*](#). Cambridge Press, 1997.

Compactness

Compactness Theorem

If every finite subset Γ_0 of Γ is satisfiable, then Γ is satisfiable.

Proof

Suppose every finite subset Γ_0 of Γ is satisfiable. By soundness, every finite subset is consistent. But since deductions are finite, it follows that Γ is consistent. Thus, by completeness, Γ is satisfiable.



Corollary

If $\Gamma \models \phi$, then for some finite $\Gamma_0 \subseteq \Gamma$ we have $\Gamma_0 \models \phi$.

Proof

Suppose to the contrary that $\Gamma_0 \not\models \phi$ for every finite $\Gamma_0 \subseteq \Gamma$. Then every finite subset of $\Gamma \cup \{\neg\phi\}$ is satisfiable, and thus $\Gamma \cup \{\neg\phi\}$ is satisfiable. It follows that $\Gamma \not\models \phi$.



Reasonable Languages

A *reasonable* language is one whose signature can be effectively enumerated and such that the two relations

$$\{ \langle P, n \rangle \mid P \text{ is an } n\text{-ary predicate symbol} \}$$

and

$$\{ \langle f, n \rangle \mid f \text{ is an } n\text{-ary function symbol} \}$$

are decidable.

Any language constructed from a finite signature is reasonable. A reasonable language must be countable.

Enumerability Theorem

Theorem

For a reasonable language, if Γ is a decidable set of formulas, then the set of all theorems of Γ is effectively enumerable.

Proof

First note that for a reasonable language, the set Λ of axioms is decidable. Given an expression, we can effectively check whether it is well-formed and whether it is a tautology or a syntactic instance of any of the other axiom groups.

Recall from lecture 2 that the set of tautological consequences of an effectively enumerable set is effectively enumerable. But as we proved earlier, ϕ is a tautological consequence of $\Gamma \cup \Lambda$ iff $\Gamma \vdash \phi$. And $\Gamma \vdash \phi$ iff $\Gamma \models \phi$ by soundness and completeness.



Corollaries

Corollary

The set of valid formulas in a reasonable language is effectively enumerable.

Corollary

If Γ is a decidable set of formulas in a reasonable language and for any sentence σ , either $\Gamma \models \sigma$ or $\Gamma \models \neg\sigma$, then the set of sentences implied by Γ is decidable.

Proof

Enumerate the theorems of Γ until either σ or $\neg\sigma$ is obtained.



Definability of a Class of Models

A *theory* is a set of sentences. For a given signature Σ , a Σ -*theory* is a set of sentences, each of which is a Σ -formula.

If $\mathcal{K}$ is a class of models with signature Σ , we say that a Σ -theory T *axiomatizes* $\mathcal{K}$ or is a *set of axioms* for $\mathcal{K}$ if $\mathcal{K}$ is the class of all models of T .

For an arbitrary Σ -theory T , let $\text{Mod } T$ be the class of all models of T (over signature Σ).

A class $\mathcal{K}$ of models is *first-order definable*, also known as an *elementary class* (EC), iff $\mathcal{K} = \text{Mod } \tau$ for some sentence τ .

A class $\mathcal{K}$ of models is *first-order axiomatizable*, or *generalized first-order definable*, also known as an *elementary class in the wider sense* (EC $_{\Delta}$), iff $\mathcal{K} = \text{Mod } T$ for some set of sentences T .

Example

Consider a signature (P) with a single binary predicate symbol P .

A model (A, R) is an *ordered set* iff R is transitive and satisfies the trichotomy condition (which states that for any $a, b \in A$ exactly one of $\langle a, b \rangle \in R$, $a = b$, $\langle b, a \rangle \in R$ holds).

This can be written as a first-order sentence as follows:

$$\begin{aligned} &\forall x \forall y \forall z (Pxy \rightarrow Pyz \rightarrow Pxz) \wedge \\ &\forall x \forall y (Pxy \vee x = y \vee Pyx) \wedge \\ &\forall x \forall y (Pxy \rightarrow \neg Pyx). \end{aligned}$$

Because these can be translated into a sentence, the class of (nonempty) ordered sets is first-order definable.

Example

Consider a signature $(\circ)$ with a single binary function symbol $\circ$.

The class of all groups is defined by the following sentence:

$$\begin{aligned} &\forall x \forall y \forall z (x \circ (y \circ z) = (x \circ y) \circ z) \wedge \\ &\forall x \forall y \exists z (x \circ z = y) \wedge \\ &\forall x \forall y \exists z (z \circ x = y). \end{aligned}$$

The class of all *infinite* groups is first-order axiomatizable. To see this, let

$$\begin{aligned} \lambda_2 &= \exists x \exists y x \neq y, \\ \lambda_3 &= \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z), \\ &\dots \\ \lambda_k &= \text{there are at least } k \text{ distinct objects} \end{aligned}$$

Then the class of infinite groups is axiomatized by the sentence for groups together with the set of sentences $\{\lambda_2, \lambda_3, \dots\}$.

Finite Models

A similar trick can be used to show the following:

Theorem

If a set Γ of sentences has arbitrarily large finite models, then it has an infinite model.

Proof

As before, for each integer $k \geq 2$, let λ_k be the sentence that translates, “there are at least k distinct objects”.

Now, consider the set $\Gamma \cup \{\lambda_2, \lambda_3, \dots\}$. By hypothesis, any finite subset has a model. So by compactness the entire set has a model, which clearly must be infinite.



Finite Models

Corollary

The class $\mathcal{K}_f$ of all finite models (for a fixed signature) is not EC_Δ , i.e. there is no set of sentences Γ such that $\mathcal{K}_f = Mod \Gamma$.

Proof

Suppose such a set Γ existed. Then since Γ has models of arbitrarily large finite size, it must also have an infinite model, which is a contradiction.

□

Corollary

The class of all infinite models is EC_Δ but not EC .

Proof

The set $\lambda_2, \lambda_3, \lambda_4, \dots$ is a first-order axiomatization of the class of all infinite models. Suppose that the class is EC . Then it is equal to $Mod \tau$ for some first-order sentence τ . But then $\mathcal{K}_f = Mod \neg \tau$ and we know that $\mathcal{K}_f$ is not even EC_Δ .

□

Finite Models

For a model M over a given signature Σ , define the *theory of M* as

$$Th\ M = \{\sigma \mid \sigma \text{ is a } \Sigma\text{-sentence which is true in } M\}.$$

Theorem

For a finite model M in a finite language, $Th\ M$ is decidable.

Proof

1. Any finite model M of size n is isomorphic to a model whose domain is $1 \dots n$.
2. We can check whether a model with domain $1 \dots n$ satisfies a sentence by building an evaluation tree which enumerates all possible assignments for each quantifier. Because the domain and the sentence are both finite, the evaluation tree will also be finite.



Finite Models

Theorem

For a finite language, $\{\sigma \mid \sigma \text{ has a finite model}\}$ is effectively enumerable.

Proof

1. Given a sentence σ and a positive integer n , we can effectively decide whether or not σ has an n -element model. This is because there are only finitely many models to check.
2. This gives us a semi-decision procedure for whether σ has a finite model. First check if it has a model of size 1, then of size 2, ...



Corollary

If Φ is the set of sentences true in every finite model, then its complement, $\overline{\Phi}$, is effectively enumerable.

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□

Corollary

If Φ is the set of sentences true in every finite model, then its complement, $\overline{\Phi}$, is effectively enumerable.

$\sigma \in \overline{\Phi}$ iff $(\neg\sigma)$ has a finite model.

Size of Models

The cardinality $|L|$ of a language L is the least infinite cardinal greater than or equal to the number of symbols in the signature of L .

The cardinality $|M|$ of a model M is the cardinality of its domain $\text{dom}(M)$.

Löwenheim-Skolem (LS) Theorem

Let Γ be a satisfiable set of formulas in a language L , then Γ is satisfiable in some model of cardinality $\kappa \leq |L|$.

Proof

By soundness, Γ is consistent, and is thus satisfiable by the model constructed in the proof of the completeness theorem. But the domain of that model is M/E which has cardinality $\leq |M|$, and $|M| = |L|$.

□

Size of Models

“Skolem’s paradox”

Let A_{ST} be your favorite set of axioms for set theory. If they are consistent, they have a model. Because the signature of the language of set theory is finite, there is a countable model. But we can prove, starting with the axioms of set theory, that there are “uncountably” many sets.

How is this possible?

Size of Models

“Skolem’s paradox”

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How is this possible?

The answer is that in the countable model of set theory, things do not correspond to what we normally think of as the model of set theory. Thus, the model of the “natural numbers” in the countable model cannot be put in one-to-one correspondence with all of the elements of the model. But this does not mean that the size of the model is truly uncountable.

Size of Models

LST Theorem

Let Γ be a set of formulas in a language of cardinality κ , and assume that Γ is satisfiable in some infinite model. Then for every cardinal $\lambda \geq \kappa$, there is a model of cardinality λ in which Γ is satisfiable.

Proof

Let M be an infinite model where Γ is satisfiable. Expand the language by adding a set C of λ new constant symbols. Let $\Delta = \{c_1 \neq c_2 \mid c_1, c_2 \text{ are distinct members of } C\}$. Then, any finite subset Γ_0 of $\Gamma \cup \Delta$ is satisfiable in M' where M' is M extended to map all constants in Γ_0 to different elements of M . By compactness, $\Gamma \cup \Delta$ is satisfiable. By the **LS Theorem**, there is a model of cardinality $\leq \lambda$. But any model must have at least cardinality λ . Thus there is a model of cardinality λ .

□

Size of Models

Corollary

If Γ is a set of sentences in a countable language, then if Γ has some infinite model, it has models of every infinite cardinality.

Two models M and M' with the same signature are *elementarily equivalent* ($M \equiv M'$) iff for any sentence σ , $\models_M \sigma$ iff $\models_{M'} \sigma$.

Corollary

If M is an infinite model for a countable language, then for any infinite cardinal λ , there is a model M' of cardinality λ such that $M \equiv M'$.

Proof

Let Γ be the set of all sentences true in M . By the corollary above, Γ has a model M' of cardinality λ . But note that for every sentence σ , either $\sigma \in \Gamma$ or $\neg\sigma \in \Gamma$ (why?). Thus, $M \equiv M'$.

□